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SIMULTANEOUS CAUSAL INFERENCE FOR MULTIPLE TREATMENTS VIA SUFFICIENCY

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Summary

Some units from a population receive the same treatment that is different from treatments available for other reservoir populations. The minimal sufficient statistic s for the pre-treatment x -covariates's distributions in the populations is the coarsest balancing score. s is used to select matching units for *simultaneous* causal comparisons of *multiple* treatments. Necessary and sufficient conditions on the posterior distribution of the treatment variable (given x) determine whether a statistic is either sufficient or minimal sufficient for the x -covariates' distributions. Results in the literature are thus extended. Strong ignorability of treatment assignment given $s(x)$ is also established. Consequently, the expected treatments' differences given $s(x)$ are shown to be simultaneously unbiased for the average causal effects of *all* treatments' differences. The existing statistical theory for s and its estimates support their use in causal inference.

Some key words: Causal Inference, Coarsest Balancing Score, Generalized Propensity Scores, Generalized Linear Models, Matching, Minimal Sufficient Statistic, Propensity Function

Running head: S-matching for Simultaneous Causal Inference

1 Introduction

When reservoir populations receive each a different treatment, the minimal sufficient statistic s of the pre-treatment x -covariates' distributions is used to select matching units for *simultaneous* causal comparisons of the treatments. Strong ignorability of treatment assignment given $s(x)$ is established and the expected treatments' differences given any s -value are shown to be *simultaneously* unbiased for the average causal effects of *all* treatments' differences. Criteria are provided to obtain either s or a balancing score from the posterior distribution, $q(t|x)$, of the treatment variable, T , given the x -covariates. The results in Imai and van Dyk (2004) are extended, providing balancing scores for a larger class of $q(t|x)$ -models.

For two treatments, $t = 1, 2$, Rosenbaum and Rubin (1983) propose the scalar propensity score $e(x)$ to balance the pre-treatment covariates, $x(\in R^p)$, of the n units in the treatment groups; $e(x)$ is the conditional probability of receiving, say, treatment 1 given x . It is stated therein that $e(x)$ is the coarsest balancing score and it is showed, among others, that if treatment assignment and the potential units' responses to treatments, $r_i(1)$ and $r_i(2)$, $i = 1, \dots, n$, are conditionally independent given x , then the difference between the sample treatments' means given $e(x)$ is unbiased for the average causal effect $E\{r(2) - r(1)\}$; E denotes expectation over the whole population.

R. Bahadur recognized that $e(x)$ is equivalent to the likelihood ratio of the x -populations' densities which is minimal sufficient (Rubin and Thomas, 1996, p. 250). As we found recently, the minimal sufficient statistic s is used with more than two treatments *a)* for dimension reduction theory, in particular when propensities do not exist (Nelson and Noorbaloochi, 2009), and *b)* in causal inference for each pair of treatments *assuming* that strong ignorability

of treatment assignment holds (Noorbalooghi, Nelson and Asgharian, 2010, p. 12, lines -2, -1, p. 13, lines 1-4). This assumption and the use of the x -covariates distributions to obtain s constitute two of the differences with this work.

For more than two treatment levels Joffe and Rosenbaum (1999) study causal effects using a small number of balancing linear functions of x . For multi-valued categorical treatments Imbens (2000) introduced the generalized propensity score $P(T = t|x)$ and used it to estimate average causal effects for treatments' pairs *but not* for simultaneous causal comparisons of all treatments; see Imbens (2000, page 709, lines 15 to 17 and lines -4 to -1). For general treatment regimes with any type of t -values, Imai and Van Dyk (2004) introduce the propensity function $e_\psi(\cdot|x)$, that has the form of $q(t|x)$, and assume that for all x -values it depends on x only through a unique, finite dimensional parameter $\theta_\psi(x)$ (*Assumption 3*, p. 856); ψ is a known parameter. They show that $e_\psi(\cdot|x)$ (i.e. $\theta_\psi(x)$) is balancing score (*Result 1*, p. 856), *not necessarily the coarsest*, and use it for simultaneous causal comparisons. Their *Assumption 3*, that does not hold when $q(t|x)$ belongs to a general exponential family, and θ_ψ 's dimensionality constitute two of the differences with this work where s is not necessarily finite-dimensional and is obtained without additional assumptions. Corollary 3.1 shows also that *Assumption 3* can be weakened to allow for more $q(t|x)$ models and *Result 1* holds automatically.

An estimate of s may not be sufficient and the same also holds for estimates of the propensity score, the generalized propensity score and the propensity function. This has been neglected so far in the Causal Inference literature that has not its own tools to confirm the balancing property unlike s ; see section 4.

The findings explain clearly what “matching” means. Units from different populations receiving each a different treatment form a matching group

when they provide the same information for the x -covariates' distributions. Such groups can be used in causal comparisons, for example, to determine the “right” dose for a new drug, by examining *simultaneously* the expected response differences $E\{r(t_2) - r(t_1)|s(x) = s_0\}$, $E\{r(t_3) - r(t_2)|s(x) = s_0\}, \dots$, $E\{(r(t_k) - r(t_{k-1})|s(x) = s_0\}$ for different doses' levels $t_1 < t_2 < \dots < t_k$.

The framework is presented in section 2. The main theoretical results and s for generalized linear models are in section 3. In section 4 some directions are given for s -matching's implementation in practice. The proofs are in the Appendix.

2 Causal inference framework and assumptions

For a random vector U use $p_U(u)$ to denote its density (but also its probability). When random vector V is also available use $p(u|v)$ to denote the conditional density of U given V . Let \mathcal{T} denote the treatments and let T be the treatment variable with values t in \mathcal{T} and prior density π_T . Treatment t is used in selected units of population \mathcal{P}_t having balanced x -covariates with respect to \mathcal{T} . The units in \mathcal{P}_t have covariates $x \in \mathcal{C}(\mathcal{P}_t) \subset R^p$ and unless otherwise stated it is assumed that $\mathcal{C}(\mathcal{P}_t) = \mathcal{C}, t \in \mathcal{T}$. Let $p(x|t)$ denote the x -covariates' density of units in \mathcal{P}_t and let $\mathcal{D}_{\mathcal{T}} = \{p(x|t), t \in \mathcal{T}\}$; $p_X(x)$ is the marginal density of the x -covariates. The notation $p(x|t)$ does not mean necessarily that p is the same density with the parameter t changing, $t \in \mathcal{T}$, but simply denotes the covariates' distribution in \mathcal{P}_t . Use $q(t|x)$ to denote T 's density (or probability) given the x -covariates. For unit i , $r_i(t)$ is the response for treatment t and the potential outcomes \mathcal{R} is the set $\{r_i(t), t \in \mathcal{T}, \text{ for } i = 1, \dots, n\}$. Conditional independence of x and y given z is denoted by $x \perp\!\!\!\perp y|z$ (Dawid,1979). The expression “covariates u, v match” means that the units with these covariates

match.

Assumption 1 (Stable Unit Treatment Value Assumption (*SUTVA*), Rubin, 1980, 1990) The distribution of potential outcomes for one unit is assumed to be independent of potential treatment status of another unit given the observed covariates.

Assumption 2 (Strong ignorability of treatment assignment given x , Rosenbaum and Rubin, 1983)

- (i) \mathcal{R} and T are conditionally independent given $x : \mathcal{R} \perp\!\!\!\perp T \mid x$, and
- (ii) for every $t \in \mathcal{T}$, $0 < p(t|x)$ (or equivalently $0 < p(x|t)$).

Recall that $b(x)$ is a balancing score if the conditional distribution of x given $b(x)$ is the same for all treatment values, i.e.

$$p(x|t, b(x)) = p(x|b(x)), \text{ for all } t \in \mathcal{T}. \quad (1)$$

From (1), thinking of t as parameter value for the distribution of x it follows that $b(x)$ is a sufficient statistic for the family $\mathcal{D}_{\mathcal{T}} = \{p(x|t); t \in \mathcal{T}\}$.

3 Matching and Causal Inference with s

In this section the minimal sufficient statistic $s(x)$ is assumed known. This is possible for various models. The results are also applicable for large samples when s is estimated.

3.1 Vector valued s

The first result, obtained directly from statistical theory, extends Nelson and Noorbaloochi (2009, p. 619, Theorem 1) and justifies the use of s for countably finite or countably infinite treatments. It also indicates that for multiple treatments s will be often vector valued.

Proposition 3.1 *Assume that \mathcal{T} consists of either countably finite or countably infinite treatments and that the covariates' distributions $\mathcal{D}_{\mathcal{T}}$ have all common support. Let $\mathcal{T}_k = \{t_1, \dots, t_k\} \subset \mathcal{T}$ and let*

$$s(x) = s^{(1)}(x)^1 = \left(\frac{p(x|t_2)}{p(x|t_1)}, \frac{p(x|t_3)}{p(x|t_1)}, \dots, \frac{p(x|t_k)}{p(x|t_1)} \right) \quad (2)$$

When s is sufficient for \mathcal{T} (i.e. for $\mathcal{D}_{\mathcal{T}}$) it is also minimal sufficient.

Remark 3.1 When $\mathcal{D}_{\mathcal{T}_k} = \{p(x|t), t \in \mathcal{T}_k\}$ do not have common support, $s(x)$'s dimension depends on the x -values (Lehmann and Casella, 1998, p. 70, Theorem 9.1).

The s -Matching Rule for covariates: *Match u to v when $s(u) = s(v)$.*

The s -matching rule can be used for any number of units using their covariates.

The next proposition shows that s -matching is not changed when

$$s^{(j)}(x) = \left(\frac{p(x|t_1)}{p(x|t_j)}, \dots, \frac{p(x|t_{j-1})}{p(x|t_j)}, \frac{p(x|t_{j+1})}{p(x|t_j)}, \dots, \frac{p(x|t_k)}{p(x|t_j)} \right), j \neq 1, \quad (3)$$

is used instead of $s = s^{(1)}$ in (2).

Proposition 3.2 *If $s(u) = s(v)$, then $s^{(j)}(u) = s^{(j)}(v)$, $j > 1$.*

Without loss of generality $s(x) = s^{(1)}(x)$ is used in this section.

Propositions 3.1 and Remark 3.1 indicate clearly that with several treatments s is not expected to be scalar. Nelson and Noorbaloochi (2009, p. 619) point out that $s(x)$ may be infinite dimensional thus contradicting the existence of a finite dimensional parameter θ in *Assumption 3* (Imai and Van Dyk, 2004). In Noorbaloochi, Nelson and Asgharian (2010, p. 8, lines 1, 2) it is also

¹In $s^{(1)}(x)$, (1) indicates the denominator is $p(x|t_1)$.

mentioned that “*In general, there is no univariate propensity score.*” This is indirectly confirmed for several data sets with high dimensional x -covariates for which the scalar propensity score deteriorates more as x ’s dimension increases (King et al., 2011, p. 18).

We revisit an example in Rosenbaum and Rubin (1983, p. 47) when the number of treatments k is larger than 2.

Example 3.1 Let $p(x|t)$ be a polynomial exponential family distribution,

$$p(x|t) = h(x) \exp\{P_t(x)\}, \quad t = 1, \dots, k,$$

with $P_t(x)$ a degree m polynomial. Then, the statistic

$$\begin{aligned} \left(\ln \frac{p(x|t_2)}{p(x|t_1)}, \dots, \ln \frac{p(x|t_k)}{p(x|t_1)} \right) &= (P_2(x) - P_1(x), \dots, P_k(x) - P_1(x)) \\ &= (Q_1(x), \dots, Q_{k-1}(x)) \end{aligned}$$

is equivalent to the minimal sufficient statistic (2) with $Q_i(x)$ a degree m polynomial, $i = 1, \dots, k - 1$.

3.2 Causal inference framework and s

To obtain s using the likelihood ratios in (2) the densities of the covariates in $\mathcal{D}_{\mathcal{T}} = \{p(x|t), t \in \mathcal{T}\}$ have either to be known or to be estimated. This may not be possible in practice. Results are now presented to determine s with the causal inference framework and without using $\mathcal{D}_{\mathcal{T}}$, simply from the conditional density $q(t|x)$ of T given the x -covariates. The first result involves $\mathcal{D}_{\mathcal{T}}$ but it is used to prove subsequent results.

Proposition 3.3 (see, e.g. Chen, 2010, Ch. 6) *Let $\mathcal{D}_{\mathcal{T}}$ be the family of the x -covariates densities. Assume that there exist function $s^*(x)$ such that for*

any covariates x_1 and x_2 the ratio $\frac{p(x_1|t)}{p(x_2|t)}$ is constant as function of t if and only if $s^*(x_1) = s^*(x_2)$. Then, $s^*(x)$ is minimal sufficient.

The tool to determine s via $q(t|x)$ is the decomposition

$$p(x|t) = q(t|x) \cdot p_X(x) \cdot \pi_T^{-1}(t), \quad (4)$$

that leads to T 's posterior factorization criterion (PFC) and the coarsest balancing score criterion (CBSC).

Proposition 3.4 *Let X and T be random vectors in Euclidean spaces with densities, respectively, p_X and p_T and with conditional densities $p(x|t)$ and $q(t|x)$. Then,*

a) (Posterior Factorization Criterion) $s(x)$ is sufficient statistic for \mathcal{D}_T if and only if

$$q(t|x) = g_1(s(x), t) \cdot g_2(t) \cdot g_3(x) \quad \forall x, t. \quad (5)$$

b) (Coarsest Balancing Score Criterion) Assume that for any x_1 and x_2 the ratio $\frac{q(t|x_1)}{q(t|x_2)}$ is independent of t if and only if $s^(x_1) = s^*(x_2)$. Then, s^* is minimal sufficient statistic for \mathcal{D}_T .*

Proposition 3.4 is used to derive directly previous results in the literature.

Example 3.2 *(The propensity score, Rosenbaum and Rubin, 1983)* The treatments $\mathcal{T} = \{1, 2\}$ and the propensity score $e(x) = q(1|x)$. For $q(t|x)$ it holds

$$q(1|x) = e(x), \quad q(2|x) = 1 - e(x). \quad (6)$$

From (6) and Proposition 3.4 a) $s(x) = e(x)$ is sufficient statistic. Since the ratio $\frac{q(t|x_1)}{q(t|x_2)}$ is independent of t for all x_1, x_2 if and only if

$$\frac{q(1|x_1)}{q(1|x_2)} = \frac{q(2|x_1)}{q(2|x_2)} \quad \Leftrightarrow \quad e(x_1) = e(x_2),$$

from Proposition 3.4 b) $e(x)$ is minimal sufficient. The same result is obtained via Proposition 3.1.

Example 3.3 (*The propensity function, Imai and van Dyk, 2004*) For various kinds of treatments $t \in \mathcal{T}$, the propensity function $e_\psi(\cdot|x) = q_\psi(\cdot|x)$ depends on x only through the unique, finite dimensional parameter $\theta_\psi(x)$ (*Assumption 3*); ψ is known parameter. Therefore, $q_\psi(t|x)$ has form (5) with $g_3(x) = 1$ for every x and from Proposition 3.4 a) $s(x) = \theta_\psi(x)$ is sufficient.

Proposition 3.4 extends the results in Imai and Van Dyk (2004) by weakening their *Assumption 3* to accommodate more $q(t|x)$ -models.

Corollary 3.1 *Let $q_\psi(t|x)$ be the density of T given the x -covariates; ψ are known parameters. Assume that there are functions $\theta_{1,\psi}(x)$, $g_{1,\psi}$, $g_{2,\psi}$, $g_{3,\psi}$ such that*

$$q_\psi(t|x) = g_{1,\psi}(\theta_{1,\psi}(x), t) \cdot g_{2,\psi}(t) \cdot g_{3,\psi}(x) \quad \forall x, t. \quad (7)$$

Then,

- a) $\theta_{1,\psi}(x)$ is a balancing score, and*
- b) $\theta_{1,\psi}(x)$ is the coarsest balancing score when for every x_1, x_2 , the ratio*

$$\frac{g_{1,\psi}(\theta_{1,\psi}(x_1), t)}{g_{1,\psi}(\theta_{1,\psi}(x_2), t)}$$

is independent of t if and only if $\theta_{1,\psi}(x_1) = \theta_{1,\psi}(x_2)$.

The minimal sufficient statistic s for \mathcal{DT} is now determined when T 's posterior is a generalized linear model.

Example 3.4 Assume the treatment variable T with values in R^d is modeled given the x -covariates ($\in R^p$) with a generalized linear model in canonical form, i.e.

$$q(t|x) = \exp\{t'b(x) + c(x) + d(t)\}; \quad (8)$$

t' denotes t 's transpose and $n \in R^d$. From Proposition 3.4 a), $b(x)$ is sufficient statistic for \mathcal{D}_T . Since the ratio

$$\frac{q(t|x_1)}{q(t|x_2)} = \exp\{t'[b(x_1) - b(x_2)] + c(x_1) - c(x_2)\}$$

is independent of t for all t, x_1, x_2 if and only if $b(x_1) = b(x_2)$, from Proposition 3.4 b) $b(x)$ is also minimal sufficient.

There are different forms $b(x)$ can have. For example, when b takes real values,

$$b(x) = \phi\left(\sum_{j=0}^p \beta_j x_j\right), \quad b(x) = \sum_{j=0}^K \beta_j \phi_j(x); \quad (9)$$

$x_0 = 1$, x_j is x 's j -th coordinate, $\beta_j \in R$, the functions ϕ, ϕ_j are assumed to be known and real valued, j 's values are according to the corresponding sum.

When $b(x)$ is known it can be used for matching units from different populations. When $b(x)$ is not known, it has to be estimated with $\hat{b}(x)$ that is used for matching.

Remark 3.2 For the generalized linear model (8) *Assumption 3* in Imai and Van Dyk (2004) does not hold because $q(t|x)$ depends on x via $c(x)$ also.

3.3 Simultaneous causal comparisons

The key result allowing for simultaneous causal comparison of several treatments follows, establishing strong ignorability of treatment assignment given $s(x)$.

Proposition 3.5 *Under Assumption 2, for the responses \mathcal{R} and the treatment variable $T = t$ it holds*

$$p\{t, \mathcal{R}|s(x) = s\} = p\{t|s(x) = s\} \cdot p\{\mathcal{R}|s(x) = s\}. \quad (10)$$

Proposition 3.5 suggests simultaneous causal comparisons using $s(x)$ to balance subpopulations for all treatments and obtain unbiased estimates of the average treatment effects.

Proposition 3.6 *Suppose that treatment assignment is strongly ignorable (Assumption 2) and that a value s_0 of $s(x)$ is randomly sampled from the population of units with covariates $x \in \mathcal{C}$. Units receiving treatments t_i and t_j are sampled with s -value for their covariates equal s_0 , $i \neq j$. Then, the expected difference in response for the units chosen is the expected treatment effect at $s(x) = s_0$. The mean of such pair differences over all $s(x)$ -values is unbiased for the average treatment effect $E\{r(t_i) - r(t_j)\}$ and the same holds, concurrently given $s(x)$, for any number of average treatment effects.*

4 Implementation

For the s -matching's implementation there are practical issues some of which depend on the data to be analyzed and the assumptions on the x -covariates models. Among these issues *a)* s should be determined, *b)* when likelihood ratios have to be estimated, the curse of dimensionality problem should be addressed, and *c)* the dimensionality of s may be reduced if there is no much loss of information. Some directions for *a)*-*c)* follow.

Known theorems in statistics (see, e.g., Lehmann and Casella, 1998) allow to obtain the minimal sufficient statistic s . Proposition 3.1 is used with a small number k of treatments to define a minimal sufficient statistic s^* for the corresponding distributions, $\mathcal{D}_{\mathcal{T}_k}$, and then show that s^* is sufficient for all the x -covariates distributions, $\mathcal{D}_{\mathcal{T}}$, i.e. $s = s^*$. For example, if $p(x|t)$ follows a normal distribution with mean t and known variance (say) 1, the minimal

sufficient statistic, s^* , is determined for the distributions $\mathcal{D}_{\{t_1, t_2\}}$; t_1, t_2 are treatments, $t_1 \neq t_2$. Neyman's factorization criterion shows that s^* is sufficient for all t -values, so it is minimal sufficient for $\mathcal{D}_{\mathcal{T}}$. Alternatively, when $p(x|t)$ belongs to a p -parameter exponential family in canonical form, Neyman's factorization criterion determines $s(x)$ that is also minimal sufficient if the parameter space \mathcal{T} contains an open, p -dimensional rectangle.

For the implementation of Proposition 3.1 in applications only a subpopulation $\tilde{\mathcal{P}}_t$ of \mathcal{P}_t may be available. Let \mathcal{MP}_t denote the units to be matched from $\tilde{\mathcal{P}}_t$ -subpopulation. Use $s = s^{(1)}$ in (2) to match a unit in \mathcal{MP}_t having covariates u with a unit from $\tilde{\mathcal{P}}_r$ having covariates $v_{m,r} \in \mathcal{C}(\tilde{\mathcal{P}}_r)$, such that

$$v_{m,r} = \arg \min_{v \in \mathcal{C}(\tilde{\mathcal{P}}_r)} \|s(u) - s(v)\|^2, \quad r \in T - \{t\}; \quad (11)$$

$\|\cdot\|$ is the usual Euclidean distance in R^p and in $v_{m,r}$ the index m denotes "matching" unit from sub-population $\tilde{\mathcal{P}}_r$. This approach is the nearest neighbor 1 : 1 matching with replacement and can be properly modified for 1 : k matching with or without replacement. For more information on matching methods and for optimal matching questions see, e.g., Rosenbaum (1989) and Stuart (2010).

Additional matching sets for \mathcal{MP}_t can be obtained using $s = s^{(j)}$ (or its estimates) in (11), $j = 2, \dots, k$, and the decision maker can select the "best" matching set, for example, that with the nearest means to the \mathcal{MP}_t covariates' means with respect to $\|\cdot\|$ or the sup-norm distance $\|\cdot\|_{\infty}$.

When the form of the x -covariates densities in $\mathcal{D}_{\mathcal{T}}$ is not known and $\mathcal{T} = \mathcal{T}_k$, the usual nonparametric estimation of each density in the ratios (2) is affected by the curse of dimensionality of the x 's. Rather than estimating separately each density one may use the approach adopted in Machine Learning for determining the ratio of the densities from the training and test data. The densities'

ratio is expressed as linear model with respect to a basis of functions. The coefficients are estimated using observations from the two populations according to a given method that usually ends with a convex minimization problem. For the description of the estimation methods in Machine Learning see, e.g., Sugiyama *et al.* (2007) and Nguyen *et al.* (2010). For independent x -samples from nonparametric models, one for each $t \in \mathcal{T}$, the class of empirical distributions is minimal sufficient.

The use of s is supported by the existing statistical theory that allows *i)* to obtain approximate sufficient statistics and evaluate the approximation's error (see, e.g., Le Cam, 1964, Joyce and Marjoram, 2008), and *ii)* use a principal components transformation of its components to investigate whether a sufficient summary of smaller dimension exists (Nelson and Noorbaloochi, 2009).

5 Appendix

Proof of Proposition 3.1: It is direct consequence of Theorem 6.12, in Lehmann and Casella, 1998, p. 37 and related theorems therein.

Proof of Proposition 3.2: Since $s(u) = s(v)$, it holds

$$\frac{p(u|t_i)}{p(u|t_1)} = \frac{p(v|t_i)}{p(v|t_1)}, \quad i = 2, \dots, k. \quad (12)$$

In (12), divide the i -th equality with the j -th equality, $i \neq j$, and invert the j -th equality to obtain

$$\frac{p(u|t_i)}{p(u|t_j)} = \frac{p(v|t_i)}{p(v|t_j)}, \quad i \neq j, \quad \text{or } s^{(j)}(u) = s^{(j)}(v).$$

Proof of Proposition 3.4: *a)* From Neyman's Factorization criterion s is sufficient statistic if and only if

$$p(x|t) = h_1(s(x), t) \cdot h_2(x) \quad \forall x, t, \quad (13)$$

and from decomposition (4)

$$\Leftrightarrow q(t|x) \cdot p_X(x) \cdot \pi_T^{-1}(t) = h_1(s(x), t) \cdot h_2(x)$$

$$\Leftrightarrow q(t|x) = h_1(s(x), t) \cdot \pi_T(t) \cdot h_2(x) \cdot p_X^{-1}(x).$$

Equality (5) follows with

$$g_1(s(x), t) = h_1(s(x), t), \quad g_2(t) = \pi_T(t), \quad g_3(x) = h_2(x) \cdot p_X^{-1}(x).$$

Conversely, from (5) Neyman's Factorization criterion (13) is obtained via (4).

b) From decomposition (4) the ratio

$$\frac{p(x_1|t)}{p(x_2|t)} = \frac{q(t|x_1)}{q(t|x_2)} \cdot \frac{p_X(x_1)}{p_X(x_2)}$$

is independent from t if and only if the ratio

$$\frac{q(t|x_1)}{q(t|x_2)}$$

is independent of t and this holds if and only if

$$s^*(x_1) = s^*(x_2).$$

Thus, from Proposition 4.1 s^* is minimal sufficient statistic.

Proof of Corollary 3.1: Both parts follow from Proposition 3.4.

Proof of Proposition 3.5: The proof follows the lines in Imai and Van Dyk (2004),

$$p\{x, t, \mathcal{R}|s(x) = s\} = p\{x, t|s(x) = s\} \cdot p\{\mathcal{R}|x, t, s(x) = s\}$$

$$\begin{aligned}
&= p\{t|s(x) = s\} \cdot p\{x|t, s(x) = s\} \cdot p\{\mathcal{R}|x, t, s(x) = s\} \\
&= p\{t|s(x) = s\} \cdot p\{x|s(x) = s\} \cdot p\{\mathcal{R}|x, s(x) = s\}.
\end{aligned}$$

The third equality is obtained using Proposition 3.1 and strong ignorability of treatment assignment given x (Assumption 2). It follows that

$$p\{t, x, \mathcal{R}|s(x) = s\} = p\{t|s(x) = s\} \cdot p\{x, \mathcal{R}|s(x) = s\}$$

Integrating both sides of the last equation over the x 's for which $s(x) = s$, we obtain that given $s(x) = s$, \mathcal{R} and T are independent.

Proof of Proposition 3.6: From Assumption 2,

$$\begin{aligned}
&E\{r(t_i)|s(x) = s, T = t_i\} - E\{r(t_j)|s(x) = s, T = t_j\} \\
&= E\{r(t_i)|s(x) = s\} - E\{r(t_j)|s(x) = s\} = E\{r(t_i) - r(t_j)|s(x) = s\}
\end{aligned}$$

and it follows that

$$E_s [E\{r(t_i) - r(t_j)|s(x) = s\}] = E\{r(t_i) - r(t_j)\};$$

E_s denotes expectation with respect to all values s of $s(x)$, $x \in \mathcal{C}$.

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